Dirac Spectrum in QCD and Quark Masses

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We use a chiral random matrix model to investigate the effects of massive quarks on the distribution of eigenvalues of QCD inspired Dirac operators. Kalkreuter's lattice analysis of the spectrum of the massive (hermitean) Dirac operator for two colors and Wilson fermions is shown to follow from a cubic equation in the quenched approximation. The quenched spectrum shows a Mott-transition from a (delocalized) Goldstone phase softly broken by the current mass, to a (localized) heavy quark phase, with quarks localized over their Compton wavelength. Both phases are distinguishable by the quark density of states at zero virtuality, with a critical quark mass of the order of 100-200 MeV. At the critical point, the quark density of states is given by $\nu_Q(\lambda) \sim |\lambda|^{1/3}$. Using Grassmannian techniques, we derive an integral representation for the resolvent of the massive Dirac operator with one-flavor in the unquenched approximation, and show that near zero virtuality, the distribution of eigenvalues is quantitatively changed by a non-zero quark mass. The generalization of our construction to arbitrary flavors is also discussed. Some recommendations for lattice simulations are suggested.

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1. Introduction

In QCD, the character of the Dirac spectrum near zero virtuality follows from symmetries alone, in the limit where the quark mass is taken to zero. This is best exemplified by the Banks-Casher relation [1] for the quark density of states, and its moments [2]. A number of recent studies have confirmed these relations using a chiral random matrix formulation, and unraveled their generic structure around zero virtuality (microscopic or mesoscopic limit) in terms of a universal spectral density [3,4].

Given the importance of the concept of current quark masses for spontaneous symmetry breaking and restoration in QCD, an important yet unanswered question in the context of chiral random matrix models, has to do with the role of a finite current quark mass. The $1/N$ expansion used in the Coulomb gas approach calls for non-trivial subleading effects [5,6]. The method of orthogonal polynomials [5] fails in the massive case [3]. Recently, Kazakov [7], Brezin, Hikami and Zee (BHZ) [8], and Zee [9] have discussed a number of alternative and powerful methods to analyze the spectral densities and level correlations of a large class of random matrix models. Their methods borrow from exact integral representations, Grassmannian formulations and diagrammatic techniques. Some of these methods will be taken up in this paper and applied to various chiral random matrix models as inspired by QCD spin and flavor symmetries. In section 2, we go over the general aspects of the spectral density, its relation to the quark condensate and its analogy with the Kubo formula. The striking violation of Lifshitz's bound [10] near zero virtuality is also discussed. In section 3, we recall results from chiral random matrix models both in the microscopic and macroscopic limit. In section 4, we analyze Kalkreuter's quenched lattice SU(2) simulations of the spectral density using recently developed arguments by Zee [9] and Kazakov [7]. We show that for a current mass $m = -(N/V_4)\langle\overline{q}q\rangle^{-1}$ (where N/V_4 is the number of quark states in the four Euclidean volume, and $\langle\overline{q}q\rangle$ the quark condensate), the spectral distribution for the unsquared and hermitean Dirac operator for Wilson fermions, shows a phase transition from a Goldstone phase to a phase where the quarks are localized over their Compton wavelength, with strong chiral symmetry breaking. The order parameter in this case is the quark density of states at zero virtuality, and the transition is reminiscent of a conductor-insulator (Mott) transition. In section 5, the unquenched problem is analyzed using the Grassmaniann method introduced by BHZ. A general integral representation for the spectral density is derived. In section 6, we show that the integral representation leads to the known spectral density in the macroscopic as well as microscopic limit for $N_F = 1$ and zero quark mass. In section 7, the case of non-zero quark mass is investigated, leading to a new spectral density in the microscopic limit and for quark masses $mN \sim 1$. The generalization to an arbitrary number of flavors is outlined. Our conclusions and recommendations are summarized in section 8.

2. QCD Spectral Distribution

The spectral representation for the quark propagator of flavor F in a fixed gluon background is given by

$$S_F(x, y, A) = \sum \frac{\phi_n(x)\phi_n^\dagger(y)}{-\lambda_n - im_F} \tag{1}$$

where ϕ_n, λ_n are eigenvectors and eigenvalues of the massless Dirac equation

$$i\nabla\!\!\!/(A)\phi_n(x) = \lambda_n\phi_n(x). \tag{2}$$

The fermion condensate in Euclidean space is

$$\langle q^\dagger q\rangle = -\sum_{F=1}^{N_F} \langle\langle\, \mathrm{Tr} S_F(x,x,A)\rangle\rangle \tag{3}$$

where $\langle\langle \ldots \rangle\rangle$ denotes the averaging over the gluonic configurations A using the QCD action with massive quarks. In the limit where the four-volume V_4 goes to infinity, the spectrum becomes dense and we may use the eigenvalue density (2)

$$\nu(\lambda, m_F) = \langle\langle \sum_n \delta(\lambda - \lambda_n)\rangle\rangle. \tag{4}$$

to characterize the Dirac spectrum. Through the fermion determinant in the averaging measure, (4) carries a non-trivial dependence on the current masses m_F. We will refer to the mass dependence in (4) as the sea mass dependence. In terms of (4) the fermion condensate (3) becomes

$$\langle q^\dagger q\rangle = \frac{1}{V_4}\sum_{F=1}^{N_F}\int d\lambda \frac{\nu(\lambda, m_F)}{\lambda + i m_F}. \tag{5}$$

The explicit mass dependence in the denominator of (5) will be referred to as the valence mass dependence, for obvious reasons. In QCD, both the sea and valence masses are the same. Here, we may choose to disentangle them (for theoretical clarity) whenever indicated. Throughout, we will think of the masses as fixed external parameters, although in QCD the quantum averaging forces them to run. This brings about the nasty issue of the ultra-violet sensitivity of (5) for finite current quark masses. Although perturbative renormalization of (5) is possible we will not discuss it here. Most of the discussions to follow, focuses on the infrared part of the spectrum (around zero virtuality). The random matrix models to be discussed below are inspired models for the constant quark modes only, with some astonishing resemblance to lattice regulated simulations.

In the chiral limit, $m_F \to 0$, the Dirac operator $i\nabla\!\!\!/$ anticommutes with γ_5, so the non-zero eigenvalues come in pairs $(\lambda, -\lambda)$ and the spectral function is symmetric. Thus

$$\langle \bar{q} q\rangle = -\frac{\pi N_F}{V_4}\nu(0). \tag{6}$$

following a Wick rotation to Minkowski space $(q^\dagger, q) \to (i\overline{q}, q)$. This relation was first derived by Banks and Casher [1]. It is important that the chiral limit is sampled with a valence quark mass m_F that is taken to zero after the thermodynamical limit $V_4 \to \infty$, for otherwise the result would be zero. The spontaneous breakdown of a continuous symmetry cannot take place in finite volumes, unless the condition $m_F\langle\overline{q}q\rangle V_4 >> 1$ is fulfilled. The result (6) states that in vector-like theories with chiral symmetric (even) spectra, the quark condensate is related to the mean spectral density at zero virtuality ($\lambda = 0$). The delocalization of the quark modes is caused by strong correlations that randomize the Dirac spectrum near zero, triggering a huge accumulation at zero. As $V_4 \to \infty$ the number of eigenvalues grows with the four volume V_4 in contrast to the length $\sqrt[4]{V_4}$ in the free case [2,11].

The change in the number of quark states near zero virtuality is drastically different from what is expected from Lifshitz's condition [10,12] for the case of scattering off random repulsive centers, where the density of states is found to vanish exponentially. The reason may be traced back to the chirality structure of the random ensemble discussed here, hence to the spin of the quarks in QCD. Lifshitz's condition can be evaded by noting that in a magnetic field further delocalization can be generated without cost of energy [11]. For spinless particles the density of states is bounded from above by the free quark density of states [1], hence no condensate is allowed to form in scalar QCD.

The Banks-Casher relation (6) is reminiscent of the conductivity in metals, where the latter is proportional to the density of states at the Fermi surface. Indeed, it follows from the Kubo formula that the d.c. conductivity σ relates to the density of states at the Fermi level, $\rho(E_F)$, through

$$\sigma = e^2\ \mathbf{D}\ \rho(E_F) \tag{7}$$

where e is the electron charge and $\mathbf{D}$ the diffusion constant. This result (7) is reminiscent of (6) with the identification of $\pi\sigma/e^2\mathbf{D}$ with $-\langle\overline{q}q\rangle$, and E_F with $\lambda \sim 0$.

3. Random Matrix Model

The pertinent random matrix model for the QCD spectrum near zero virtuality follows from the color representation of the quark fields (here fundamental), the chirality odd character of the massless Dirac operator in a fixed gluon background and the number of flavors. In the continuum, the QCD Dirac operator for three colors may be mapped onto the Gaussian Unitary Ensemble (GUE), while for two colors it may be mapped onto the Gaussian Othogonal Ensemble (GOE) [4]. In the zero topological charge sector, and for three colors the generating functional is [3,4,13–15],

$$Z[m] = \int\ dT\ \prod_F^{N_F} \det\begin{pmatrix} im_F & T \\ T^\dagger & im_F\end{pmatrix}\ e^{-N\Sigma^2\,\mathrm{Tr}(T^\dagger T)} \tag{8}$$

where T is a random $N\times N$ complex matrix, with N identified with the four volume V_4 and Σ the chiral condensate appearing in the Banks-Casher relation (6). Equation (8) is the generating function for the GUE. The joint eigenvalue density following from (8) reads [3]

$$\nu(\lambda_1, ..., \lambda_N; m_F) = \mathbf{C}_N \prod_{i\leq j} |\lambda_i^2 - \lambda_j^2|^2$$

$$\prod_i (\lambda_i^2 + m_F^2)^{N_F} |\lambda_i| \, e^{-N\Sigma^2 \sum_{l=1}^N \lambda_l^2}. \quad (9)$$

with $\mathbf{C}_N$ an overall normalization (see below). Integrating (9) over $(N-1)$-eigenvalues and taking the microscopic limit $N \to \infty$ with $x = N\lambda$ fixed,[1] yields the following form for the microscopic spectral density ($m_F = 0$) [3]

$$\nu_s(x) = 2\Sigma^2 x \left(J_{N_F}^2(2\Sigma x) - J_{N_F-1}(2\Sigma x) J_{N_F+1}(2\Sigma x) \right) \quad (10)$$

which is the master formula for all the sum rules discussed by Leutwyler and Smilga [2]. Equation (10) shows that around zero virtuality ($\lambda \sim 0$), the distribution of eigenvalues oscillates to zero. For $N_F = 0$ these oscillations are caused by the level repulsion around zero due to the symmetric character of the spectrum under chirality (Airy phenomenon [8]). These oscillations are affected by the fermionic determinant in the massless case, as is clear from the N_F dependence (zero mode suppression). In the quenched approximation, the character of these oscillations appear to be unaffected by the choice of the measure, provided that it is local. Non-local changes to the measure, *e.g.* through a fermion determinant, do affect the structure of these oscillations. Other non-local changes are also possible, but will not be discussed in this work.

The joint eigenvalue density associated to (9) can be obtained through a direct integration, or by using a Coulomb gas description of (8). Indeed, using the definition (4) for $\nu(\lambda)$, where the average is over the joint eigenvalue density (9), allows a rewriting of (8) in terms of an effective action

$$\begin{aligned}\mathbf{S}[\nu] = &-\int d\lambda d\lambda' \, \nu(\lambda)\nu(\lambda') \ln|\lambda^2 - \lambda'^2| \\ &-\int d\nu \, \nu(\lambda) \left(\sum_{F=1}^{N_F} \ln(\lambda^2 + m_F^2) + \ln|\lambda| \right) \\ &+N\Sigma^2 \int \nu(\lambda) \, \lambda^2 \\ &+\xi \left(\int d\lambda \, \nu(\lambda) - N \right)\end{aligned} \quad (11)$$

where ξ is a Lagrange multiplier. For a dense spectrum, the integration over the eigenvalues λ_i may be traded by a functional integration over the eigenvalue density $\nu(\lambda)$, modulo a Jacobian [6]. In the large N limit, the extremum of $\mathbf{S}[\nu]$ including the contribution from the Jacobian to the effective action, determines the macroscopic spectral density. Variation of (11) with respect to ν and differentiation with respect to λ, yield

$$2\mathbf{P} \int \frac{\nu(\lambda')}{\lambda^2 - \lambda'^2} = N\Sigma^2 - \sum_{F=1}^{N_F} \frac{1}{\lambda^2 + m_F^2} - \frac{1}{2\lambda^2} + \mathbf{J} \quad (12)$$

with the normalization condition

$$\int d\lambda \, \nu(\lambda) = N \quad (13)$$

The contribution $\mathbf{J}$ is due to the Jacobian and is of order $1/N^2$ [6]. Its explicit form will not be needed here. In (12), $\mathbf{P}$ stands for the principal value of the integral. In the thermodynamical limit ($N \to \infty$), the fermions and the Jacobian drop from the macroscopic spectral density in the chiral limit. Thus

$$\mathbf{P} \int \frac{\nu(\lambda')}{\lambda^2 - \lambda'^2} = N \frac{\Sigma^2}{2} \quad (14)$$

This integral equation yields a semi-circular distribution for the macroscopic spectral density

$$\nu(\lambda) = \frac{N\Sigma}{\pi} \sqrt{1 - \frac{\lambda^2 \Sigma^2}{4}} \quad (15)$$

The level repulsion revealed in the microscopic limit takes place in a window of size $1/N$ around the origin, and shrinks to zero size in the thermodynamical limit. In terms of (15), the quark condensate is $\langle \overline{q} q \rangle = -N_F(N/V_4)\Sigma$. Here N/V_4 is just the eigenvalue density of the Dirac operator in Euclidean space. Throughout Σ and V_4 are set to one, and the thermodynamical limit is understood for $N \to \infty$. The scale Σ can be reinstated at the end by inspection. Since the thermodynamical limits are now understood, we prefer to work from this moment with the macroscopic spectral density (15) normalized to one instead of N, *i.e.*,

$$\nu(\lambda) = \frac{1}{2\pi} \sqrt{4 - \lambda^2}. \quad (16)$$

unless specified otherwise.

4. Quenched Spectral Distributions

Recent detailed numerical analysis by Kalkreuter [16] using Wilson as well as staggered fermions, provides some useful insights to the macroscopic character of the fermionic spectrum of four-dimensional gauge theories. On the lattice, the spin-Lorentz structure of the QCD Dirac operator is affected. Indeed, for any number of colors greater than two[2] and Wilson fermions the pertinent random matrix ensemble is the Gaussian Unitary

[1]Note that our definition differs slightly from the one used in reference [3] due to replacement $2N \leftrightarrow N$, therefore $\nu_s(x) \leftrightarrow 2\nu_s(2x)$.

[2]For two colors the ensemble is the Gaussian Orthogonal Ensemble (GOE).

Ensemble (GUE). The chiral structure is upset by Wilson's r-terms, needed to remove the lattice doublers. For two colors and staggered fermions the chirality structure is preserved, but the Lorentz structure is upset. The pertinent random matrix ensemble is the Gaussian Symplectic Ensemble (GSE) [4]. For three colors it is back to the GUE.

- Wilson Fermions

In the quenched approximation, Kalkreuter's results for $SU(2)_c$ with Wilson fermions for the unsquared and unnormalized operator $\mathbf{Q} = \gamma_5(\nabla\!\!\!/ + m)$ (continuum), may be mocked up by the Gaussian Orthogonal Ensemble (GOE), provided that the gauge configurations are sufficiently random. Specifically [3],

$$\mathbf{Q}_W = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} + \mathbf{R} \tag{17}$$

which is the sum of a deterministic $2N \times 2N$ matrix (first contribution, with m - diagonal block $N \times N$), and a random hermitean (symmetric) $2N \times 2N$ matrix. We note that $\mathbf{Q}_W^\dagger = \mathbf{Q}_W$. The measure is

$$\mathbf{P}(\mathbf{R}) = \frac{1}{Z}\, e^{-2N\mathrm{Tr}(\mathbf{R}\mathbf{R}^\dagger)} \tag{18}$$

This problem is reminiscent of an electron scattering on impurities in a spin dependent quantum Hall fluid, as recently suggested by Zee [9].

The distribution of eigenvalues of $\mathbf{Q}_W$

$$\nu_Q(\lambda, m) = \frac{1}{2N} < \mathrm{Tr}_{2N}\, \delta\,(\lambda - \mathbf{Q}_W) > \tag{19}$$

follows from the resolvent $\mathbf{G}(z, m)$ of $\mathbf{Q}_W$ through

$$\nu_Q(\lambda, m) = -\frac{1}{\pi}\mathrm{Im}\ \mathbf{G}(\lambda + i0, m) \tag{20}$$

Since $\mathbf{R}$ is only hermitean (symmetric), the distribution of eigenvalues is only symmetric about zero virtuality on the average. In the chiral limit, the states are not necessarily paired about zero, because of the r-terms in the Wilson action.

The problem of determining the one-point Green function $\mathbf{G}(z)$ for the sum of a deterministic Hamiltonian H_0 with eigenvalues $\epsilon_i, (\,i = 1, \ldots, N)$ and a Gaussian random matrix was first solved by Pastur [17], and recently rederived and generalized using much simpler arguments by Zee [9]. Generally, the Green function is determined from

$$\mathbf{G}(z) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{z - \epsilon_i - \mathbf{G}(z)}. \tag{21}$$

[3]For one flavor we set $m_F = m$.

For the case (17), the deterministic hamiltonian is composed of two diagonal blocks with eigenvalues $\pm m$, so $\mathbf{G}(z, m)$ satisfies the cubic equation

$$\mathbf{G}^3 - 2z\mathbf{G}^2 + (z^2 - m^2 + 1)\mathbf{G} - z = 0 \tag{22}$$

Using Cardano's complex solution to (22) we get

$$\nu_Q(\lambda, m) = \frac{\sqrt{3}}{2}\left[(r + \sqrt{d})^{1/3} - (r - \sqrt{d})^{1/3}\right] \tag{23}$$

with

$$r = \frac{1}{6}\lambda\left(1 + 2m^2 - \frac{2}{9}\lambda^2\right)$$
$$d = \frac{1}{27}\left[(1 - m^2)^3 - \lambda^2(\frac{1}{4} - 5m^2 - 2m^4 + m^2\lambda^2)\right] \tag{24}$$

At the critical point $m_* \sim 1$ and near zero virtuality $\lambda \sim 0$, the order parameter behaves as $\nu_Q(\lambda, 1) \sim |\lambda|^{1/3}$, with the critical exponent $\beta = 1/3$. This is to be contrasted with the density of states of electrons near the mobility edge, rescattering off random impurities and undergoing Anderson localization with $\beta = 0$ [18–20].

It is worth mentioning, that although the random matrix model under consideration is only valid for the constant quark modes, it does reproduce the bulk characteristic of the spectral function from lattice regulated calculations. This justifies *a posteriori* our assumption in ignoring the ultraviolet aspects of the quark spectrum, with their inherent diverging contribution to the quark condensate. We suspect that a cooling of Kalkreuter's gauge configurations will not affect considerably the character of the spectral distribution. Such procedure can be used to define unambiguously the quark condensate for finite current masses.

The distribution of eigenvalues of $\mathbf{Q}_W$ is strikingly similar to the one discussed recently in the finite temperature problem for the hermitean and massless Dirac operator $i\nabla\!\!\!/$ on the torus in the quenched approximation [21–23]. This may be understood if we note that in the high temperature phase, dimensional reduction implies that in Euclidean space and in one-dimension lower, the effective quark mass is not m but $\sqrt{m^2 + \pi^2 T^2}$ after suitable chiral rotations [24]. In three dimensions the random ensemble is indeed the unitary ensemble [25]. This observation can be used to map out the chiral sensitive part of the QCD phase diagram with Wilson fermions in the plane spanned by the temperature and current quark mass. We note that in the context of phase transitions, and if $\mathbf{G}$ (or at least its discontinuity along the real axis) is understood as an order parameter, then the cubic equation (22) is generic of second order phase transitions, as expected from universality arguments. Some of these issues will be discussed elsewhere.

Kalkreuter's results are displayed in Fig. 1 for $\beta = 4/g^2 = 0$ (strong coupling) and different values of $\kappa = (2m + 8)^{-1}$ (Wilson fermions). A similar, although

weaker, behaviour is also seen for $\beta = 1.8$ (weak coupling) and for a range of κ's close to the critical value of 0.125 [26].

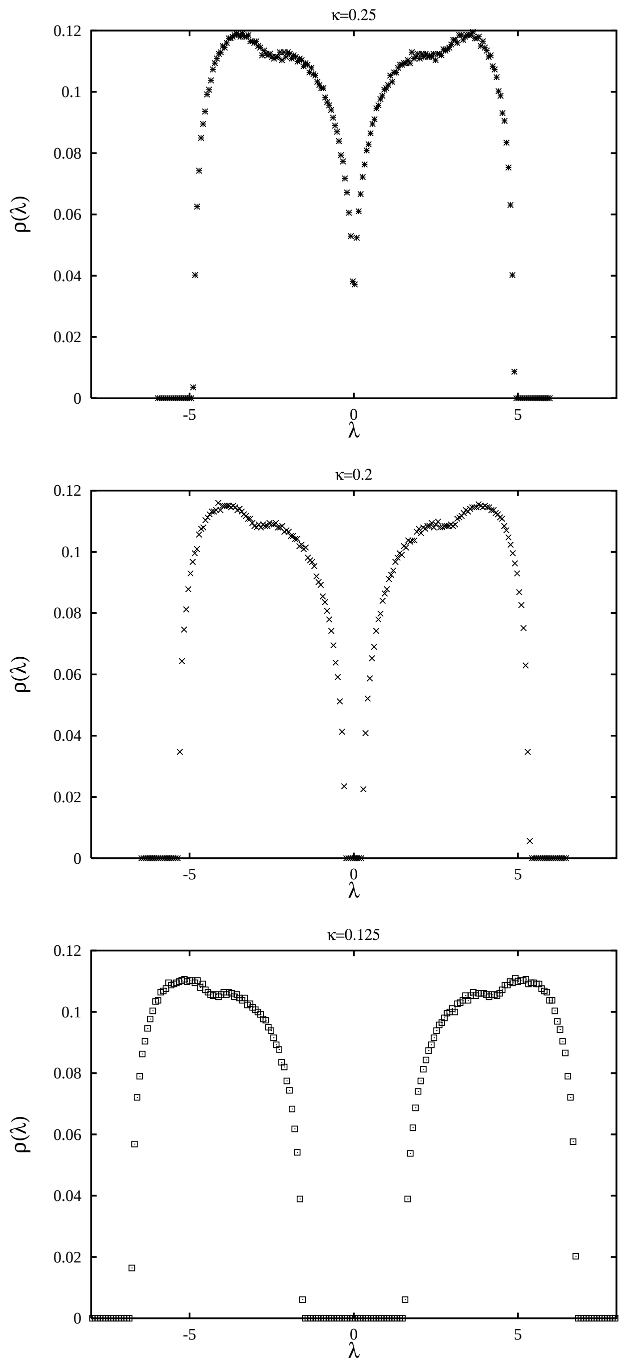

FIG. 1. Spectral densities for the operator Q_W for Wilson fermions at $\beta = 0$, obtained by Kalkreuter [16]. The stars (upper), crosses (middle), squares (lower) correspond to the values of κ equal to 0.25 ,0.20, 0.125, respectively.

The behavior of (23) versus λ for different values m is shown in Fig. 2.

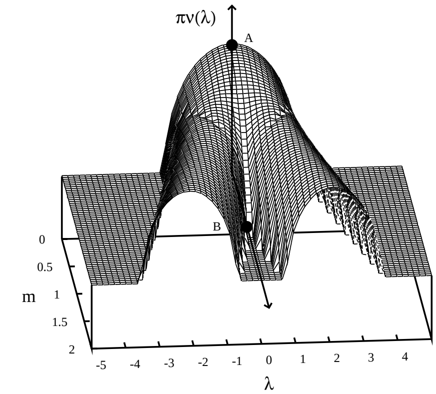

FIG. 2. Spectral function $\pi\nu_Q(\lambda, m)$ (23) as a function of the mass m and eigenvalue λ. Point A fixes the normalization for the massless case, point B corresponds to the critical mass $m_* = 1$. All axes are scaled in units of $\Sigma = 1$.

In the massless case, (23) reduces to (15) in units of $1/\Sigma$, with a half-width equal to 2. In the limit $m >> 2$, the density of states decompose into two pieces, with $\nu_Q(0, m) = 0$, and a width of the order of $3/\sqrt{2}$ [9], which is to be contrasted with 4, the size of the semi-circle. This is a regime, where chiral symmetry is strongly broken by a massive quark. For $m_* = 1$, the two regions merge into each other. In physical units, the spectral transition occurs for quark masses of the order of $m_* \sim -(N/V_4)\,\langle\overline{u}u\rangle^{-1}$. Typically $N/V_4 \sim 1$ fm^{-4}, and for a quark condensate in the range $(200MeV)^3$ - $(250MeV)^3$ we get a critical mass in the range $m_* \sim 100 - 200$ MeV, which is rather close to the strange quark mass. The spectral transition characterizes the transition from a delocalized phase with softly broken chiral symmetry (Goldstone phase) to a localized phase with strongly broken chiral symmetry (heavy quark phase). The transition is characterized by the quark density of states at zero virtuality, $\nu(\lambda \sim 0, m)$. It is non-zero in the Goldstone phase, and zero in the heavy-quark phase. This transition is reminiscent of a Mott-transition from a conductor (Goldstone phase) to an insulator (heavy quark phase).

- Staggered Fermions

For staggered fermions and two colors, it is more appropriate to use the Gaussian Symplectic Ensemble (GSE) [4], while for three colors the GUE. For illustration, consider the GUE for three colors, that is

$$\mathbf{Q}_S = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} + \begin{pmatrix} 0 & -iT \\ iT^\dagger & 0 \end{pmatrix} \tag{25}$$

which is again the sum of a deterministic $2N \times 2N$ matrix (first contribution, with m - diagonal block $N \times N$), and a random hermitean complex $2N \times 2N$ matrix (second contribution, with T - block $N \times N$). The resolvent associated to (25) is

$$\mathbf{G}(z) = \frac{z}{2}\left(1 - i\frac{\sqrt{4+m^2-z^2}}{\sqrt{z^2-m^2}}\right) \tag{26}$$

and the corresponding macroscopic spectral distribution is

$$\nu_Q(\lambda, m) = \frac{|\lambda|}{2\pi}\frac{\sqrt{4+m^2-\lambda^2}}{\sqrt{\lambda^2-m^2}} \times\Theta(\lambda^2-m^2)\;\Theta(4+m^2-\lambda^2) \tag{27}$$

where Θ are step functions. The macroscopic spectral distribution is symmetric about the origin, with a support from m to $\sqrt{4-m^2}$ to the right, and $-\sqrt{4-m^2}$ and $-m$ to the left. The appearance of the gap at zero virtuality is remiscent of the familiar mass gap in the Dirac equation. Indeed, $\mathbf{Q}_S$ is nothingh but the Hamiltonian of a quark in a five-dimensional space where γ_5 plays the role of the fifth gamma matrix along the extra (temporal) direction.

The spectral distribution for the squared staggered operator $\mathbf{Q}_S^2$,

$$\mathbf{Q}_S^2 = \begin{pmatrix} m^2+TT^\dagger & 0 \\ 0 & m^2+T^\dagger T\end{pmatrix} \tag{28}$$

that is

$$\nu_{Q^2}(\lambda^2, m) = \frac{1}{2N} < \mathrm{Tr}_{2N}(\lambda^2-\mathbf{Q}_S^2) > \tag{29}$$

with the averaging carried out using (18), can be readily tied to (27) through

$$\nu_{Q^2}(\lambda^2, m) = \frac{1}{|\lambda|}\;\nu_Q(\lambda, m) \tag{30}$$

This result can also be checked to follow from the large N analysis using the method suggested by Kazakov or BHZ.

To show this, consider the modified density of states

$$\tilde{\nu}(\lambda, m) = \sqrt{\lambda^2-m^2} << \mathrm{Tr}_N\delta(\lambda^2-m^2-T^\dagger T) >> \tag{31}$$

which is seen to reduce to (15) in the massless case, $\tilde{\nu}(\lambda, 0) = \nu(\lambda)$. Note that the normalization is now to N. In terms of (31), the quark condensate reads

$$\langle q^\dagger q\rangle = -i\int\frac{m}{|\lambda|}\frac{1}{\sqrt{\lambda^2-m^2}}\,\tilde{\nu}(\lambda, m)\,d\lambda \tag{32}$$

Following Kazakov [7], we modify the Gaussian probability distribution (18) by adding an auxiliary matrix source A,

$$\mathbf{P}_A(T) = \frac{1}{Z_A}e^{-N\mathrm{Tr}\,(T^\dagger T - AT^\dagger T)} \tag{33}$$

As a result, the Fourier transform of the resolvent reads

$$U_A(\tau) = \sqrt{\lambda^2-m^2}\left\langle\frac{1}{N}\mathrm{Tr}e^{i\tau T^\dagger T + i\tau m^2}\right\rangle_A \tag{34}$$

where the subscript A denotes averaging with the modified probability distribution (33). In terms of (34), the spectral density is

$$\nu(\lambda, m) = \sqrt{\lambda^2-m^2}\int_{-\infty}^{+\infty}\frac{d\tau}{2\pi}e^{i\tau\lambda^2}U_{A=0}(\tau) \tag{35}$$

where A is set to zero, only after the averaging in (34) is carried out. The result for the macroscopic spectral density is

$$\tilde{\nu}(\lambda, m) = \frac{1}{2\pi}\sqrt{4+m^2-\lambda^2} \times\Theta(\lambda^2-m^2)\;\Theta(4+m^2-\lambda^2) \tag{36}$$

in agreement with (30-31).

The above arguments are subtle at the edge of the spectral distributions where Airy oscillations are expected [8]. These oscillations are at the origin of (10) in the quenched approximation. To analyze them in the staggered case, we need to magnify the edge points at the level of one level spacing. This can be achieved for instance, by taking the microscopic limit $N\to\infty$ with $N\sqrt{\lambda^2-m^2}$ fixed. For the chiral unitary ensemble, the corresponding microscopic spectral density reads (quenched approximation)

$$\tilde{\nu}_s(\lambda, m) = \frac{N}{2}\sqrt{\lambda^2-m^2} \times\left(J_0^2(2N\sqrt{\lambda^2-m^2}) + J_1^2(2N\sqrt{\lambda^2-m^2})\right) \tag{37}$$

In the region $\pm m$, the spectral density dives to zero again, at a rate that is comparable to the one discussed in the massless case (10). The effects of the mass on the unquenched spectral density near zero virtuality are more difficult to track down with the methods described above. They may be analyzed by Grassmannian techniques as we now discuss.

5. Integral Representation : $N_F = 1$

In a recent analysis, BHZ have put forward an alternative method to the Coulomb gas approach and the orthogonal polynomial construction to study the non-Gaussian character of the density of eigenvalues. Their method uses Grassmannian techniques, and proved to be very elegant to get at the oscillating character of the density of states near zero virtuality. Here, we will use their method to analyze the elusive case of massive, unquenched QCD with three colors. Using the GUE, we will discuss the one flavor case in details, and outline the generalization to the case of two and more flavors.

The density of eigenvalues $\nu(\lambda, m)$ will be again sought in terms of the discontinuity of the resolvent $G(z, m)$ along the real axis. The latter is now given by

$$\mathbf{G}(z, m) = \frac{1}{2N}\left\langle\mathrm{Tr}_{2N}\left(\frac{1}{z-\mathbf{M}}\right)\right\rangle \tag{38}$$

with

$$\mathbf{M} = \begin{pmatrix} 0 & T \\ T^\dagger & 0 \end{pmatrix} \tag{39}$$

The averaging in (38) is over the distribution $\mathbf{P}(T)$ for complex $N \times N$ matrices T

$$\mathbf{P}(T) = \frac{1}{Z} e^{-N\mathrm{Tr}(TT^\dagger)} \prod_{F=1}^{N_F} \det{}_{2N} \mathbf{M}_F \tag{40}$$

where Z is an overall normalization, including the fermion determinant. For each flavor,

$$\mathbf{M}_F = \begin{pmatrix} im & T \\ T^\dagger & im \end{pmatrix} \tag{41}$$

First, let us consider the case of only one flavor $N_F = 1$. Following the method discussed by BHZ, we can write the unaveraged contribution to the resolvent as follows

$$\begin{aligned} &\frac{1}{2N}\left(\mathrm{Tr}_{2N}\frac{1}{z-\mathbf{M}}\right) \det{}_{2N} \mathbf{M}_1 \\ &= \int d\mu\,(\langle a|a\rangle + \langle b|b\rangle) \\ &\times e^{iNz(\langle a|a\rangle + \langle b|b\rangle + \langle\alpha|\alpha\rangle + \langle\beta|\beta\rangle)} \\ &\times e^{-iN(\langle a|T^\dagger|b\rangle + \langle\alpha|T^\dagger|\beta\rangle - \langle\mu|T^\dagger|\nu\rangle + \text{h.c.})} \\ &\times e^{-Nm(\langle\mu|\mu\rangle + \langle\nu|\nu\rangle)} \end{aligned} \tag{42}$$

where a_i, b_i are N-dimensional complex vectors, and $\alpha_i, \beta_i, \mu_i, \nu_i$ are N-dimensional Grassmannian vectors. The measure $d\mu$ is graded,

$$d\mu = \prod_{i=1}^{N} [da_i][db_i][d\alpha_i][d\beta_i][d\mu_i][d\nu_i] \tag{43}$$

Averaging (42) over T using the Gaussian measure (40) gives,

$$\begin{aligned} \mathbf{G}(z,m) = \frac{1}{Z}\int \prod_{i=1}^{N} d\mu\,(\langle a|a\rangle + \langle b|b\rangle) \\ e^{iNz\,(\langle a|a\rangle + \langle b|b\rangle + \langle\alpha|\alpha\rangle + \langle\beta|\beta\rangle)} \\ \times e^{-Nm(\langle\mu|\mu\rangle + \langle\nu|\nu\rangle) - N\mathrm{Tr_N}} \end{aligned} \tag{44}$$

with

$$\begin{aligned} \mathrm{Tr}_N = \mathrm{Tr}_N((-|b\rangle\langle a| + |\beta\rangle\langle\alpha| + |\nu\rangle\langle\mu|) \\ \otimes(-|a\rangle\langle b| + |\alpha\rangle\langle\beta| + |\mu\rangle\langle\nu|)) \end{aligned} \tag{45}$$

The tensor product in (44) generates four Fermi fields bilinears. They can be linearized by introducing four auxiliary complex fields σ_{ij} with $i,j = 1,2$, *e.g.*

$$\begin{aligned} &e^{N\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle} = \\ &+\frac{N}{\pi}\int d^2\sigma_{11} e^{-N(\sigma_{11}^*\sigma_{11} + \sigma_{11}^*\langle\alpha|\alpha\rangle + \sigma_{11}\langle\beta|\beta\rangle)} \end{aligned} \tag{46}$$

and so on. In terms of (44), the Grassmannian integrations can be undone, and the result is

$$\begin{aligned} \mathbf{G}(z) \propto \int \prod_{i=1}^{N} [da_i][db_i][d\sigma](\langle a|a\rangle + \langle b|b\rangle) \\ \times e^{iNz(\langle a|a\rangle + \langle b|b\rangle) - N\langle a|a\rangle\langle b|b\rangle} \\ \times \det\mathbf{\Delta} \end{aligned} \tag{47}$$

where $4N \times 4N$ block matrix reads

$$\mathbf{\Delta} = \begin{pmatrix} s_1^* & |a\rangle\langle b| & s_3^* & 0 \\ |b\rangle\langle a| & s_1 & 0 & s_4 \\ s_4^* & 0 & s_2^* & |a\rangle\langle b| \\ 0 & s_3 & |b\rangle\langle a| & s_2 \end{pmatrix} \tag{48}$$

with $s_1 = iz - \sigma_{11}$, $s_2 = m - \sigma_{22}$, $s_3 = -\sigma_{12}$ and $s_4 = -\sigma_{21}$. The matrix (48) is sparse. It can be rotated to an N block-diagonal form of 4×4 matrices. Hence

$$\begin{aligned} \det\mathbf{\Delta} = ((a^2b^2)^2 - a^2b^2\sum_{i=1}^{4} s_i s_i^* + |s_1s_2 - s_3s_4|^2)) \\ \times\left(|s_1s_2 - s_3s_4|^2\right)^{N-1} \end{aligned} \tag{49}$$

with $a^2 = \langle a|a\rangle$ and $b^2 = \langle b|b\rangle$. The integrals in (48) may be simplified by introducing the new variables

$$s_{ij} = \sigma_{ij} - iz\delta_{11} - m\delta_{22}, \tag{50}$$

$$s_{ij}^* = \sigma_{ij}^* - iz\delta_{11} - m\delta_{22}. \tag{51}$$

and $x = a^2$ and $y = b^2$. δ_{11} is non-zero for $i = j = 1$ and similarly for δ_{22}. Thanks to the Gaussian term in the integration over σ and $z = \lambda + i\epsilon$, we can freely shift the contour of integration, in particular we can treat matrices $\mathbf{s}$ and $\mathbf{s}^*$ as complex conjugate. Hence

$$\begin{aligned} G(z,m) = \mathcal{N} e^{-Nm^2}\int_0^\infty dx\; dy\; ds_{ij}\; ds_{ij}^* \\ \times\,(x+y)(xy)^{N-1} e^{-Nxy - N\mathrm{tr}ss^\dagger + Nm(s_{22}+s_{22}^*)} \\ \times\, e^{i\zeta(x+y-s_{11}-s_{11}^*)} e^{\zeta^2/N} \det\Delta(ss^\dagger, xy) \end{aligned} \tag{52}$$

where $\mathcal{N}$ is a normalization constant and

$$\det\mathbf{\Delta} = (x^2y^2 - xy\mathrm{tr}\mathbf{s}\mathbf{s}^\dagger + \det\mathbf{s}\,\mathbf{s}^\dagger)(\det\mathbf{s}\,\mathbf{s}^\dagger)^{N-1}. \tag{53}$$

Notice that the saddle point in s_{ij} has a symmetry with respect to the unitary rotations of $\mathbf{s}$ for $m = 0$. Only the $\mathcal{O}(1)$ term breaks this symmetry. The relations (52,53) constitute the integral representation for the resolvent $\mathbf{G}(z,m)$ in the complex z-plane, for one massive flavor and finite dimension N. Its singularity along the real axis $z = \lambda + i\epsilon$ corresponds to the distribution of eigenvalues of the QCD Dirac operator with one massive flavor as modeled by the chiral random matrix ensemble.

In the thermodynamical limit $N \to \infty$, the integral in (52) is dominated by the saddle point configurations. The latter are the extrema of

$$\mathbf{S}_{\rm eff} = xy - \ln\, xy - iz(x+y) \\ +{\rm Tr}(\mathbf{s}\,\mathbf{s}^\dagger) - \ln({\rm det}\mathbf{s}\,\mathbf{s}^\dagger) \tag{54}$$

The extremum of (54) coincides with the extremum in the quenched approximation (BHZ), leading to a semi-circular distribution of eigenvalues. The fermion determinant does not affect the eigenvalue distributions in the large N limit. As observed in [3] and discussed by BHZ, the saddle point approximation breaks down at $\lambda = 0$ and also the edges of the semi-circular distribution due to Airy-type singularities (quenched) and zero modes (unquenched).

6. Microscopic Limit : Massless Case

To investigate the behavior of the spectral density near the origin for the GUE, we will use the microscopic limit discussed in [3]. We will simply blow up the spectrum near $\lambda = 0$ over a scale of the order of one-level spacing. This is achieved by considering $Nz = \zeta = \mathcal{O}(1)$ in the region $z = 0$ (zero virtuality), and track down the $1/N$ corrections. Since the mass term and the $1/N$ corrections violate the saddle point invariance under unitary rotations, care is to be used to disentangle the soft directions from the hard directions. All soft directions have to be integrated out exactly, for an accurate estimate of the saddle around $z = 0$.

First, let us consider the massless case $m = 0$, where only the $\mathcal{O}(1)$ terms break the unitary symmetry of $\mathbf{s}$. To make the latter manifest, we use the parameterization as

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{u} & 0 \\ \sqrt{v}e^{i\beta} & \sqrt{w}e^{i\gamma} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ \times \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{i\psi} \end{pmatrix}. \tag{55}$$

so that

$$\begin{aligned} {\rm tr} ss^\dagger &= u+v+w, & (56)\\ \det ss^\dagger &= uw & (57)\\ s_{11}+s_{11}^* &= 2\sqrt{u}\cos\theta\cos\phi. & (58)\end{aligned}$$

Following BHZ let us introduce also

$$\begin{aligned} xy &= p, & (59)\\ x+y &= 2\sqrt{p}\cosh\omega. & (60)\end{aligned}$$

With these notations we have neglecting the $\mathcal{O}(1/N)$ term in the exponent

$$G(z,0) = \mathcal{N}' \int_0^\infty \cosh\omega d\omega \int_0^\infty \frac{dp}{\sqrt{p}} \\ \times \int_0^\infty u du dv dw \int_0^{2\pi} d\phi d\psi d\beta d\gamma \tag{61} \\ \times \int_0^{\pi/2} \sin\theta\cos\theta d\theta p^N (uw)^{N-1}$$

$$\times\, (p^2 - p(u+w+v) + uw) \tag{62}$$

$$\times\, e^{-N(u+w+v+p)} \\ \times\, e^{2i\zeta\sqrt{p}\cosh\omega} e^{2i\zeta\sqrt{u}\cos\phi\cos\theta}. \tag{63}$$

The ψ, β and γ angular integrals can be trivially performed. Also the v integral is trivial. The θ and ϕ integrals can be performed using

$$\int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta\; \sin\theta\; \cos\theta e^{-i\zeta\sqrt{u}\cos\theta\cos\phi} \\ = 2\pi \frac{J_1(2\zeta\sqrt{u})}{2\zeta\sqrt{u}} \tag{64}$$

and the ω integral by

$$\int_0^\infty d\omega \cosh\omega e^{2i\zeta\sqrt{p}\cosh\omega} = K_1(-2i\zeta\sqrt{p}) \tag{65} \\ = -\frac{\pi}{2}(J_1(2\zeta\sqrt{p}) + iN_1(2\zeta\sqrt{p})).$$

For the imaginary part of $G(\zeta, 0)$ we get

$$\rho(\zeta/N, 0) = -\pi^2\sqrt{2}\left(\frac{N}{\pi}\right)^{7/2} e^{3N} \int_0^\infty \frac{dp}{\sqrt{p}} du \frac{dw}{w} \\ \times\, (p^2 - p(u+w+\frac{1}{N}) + uw) \\ \times\, (puw)^N e^{-N(p+u+w)} J_1(2\zeta\sqrt{p}) \frac{J_1(2\zeta\sqrt{u})}{2\zeta\sqrt{u}}. \tag{66}$$

Where we have explicitly wrote down the normalization (we used Stirling formula to rewrite the normalization in this form). This integral is dominated by the neighborhood of the saddle point

$$p_c = u_c = w_c = 1. \tag{67}$$

Writing

$$\begin{aligned} p &= 1 + \frac{p'}{\sqrt{N}}, & (68)\\ u &= 1 + \frac{u'}{\sqrt{N}}, \\ w &= 1 + \frac{w'}{\sqrt{N}}, \end{aligned}$$

we have

$$p^2 - p(u+w+1/N) + uw = \\ \frac{1}{N}((p')^2 - 1 - u'p' + w'(u'-p') - \frac{1}{\sqrt{N}}p'). \tag{69}$$

The leading contribution coming from $(p')^2 - 1$ cancels out. We are therefore led to consider the next order terms. We observe that due to the symmetry in p' and u' the terms containing w' cancel. Using (69) we have

$$\frac{J_1(2\zeta\sqrt{u})}{\sqrt{u}}$$
$$= J_1(2\zeta) - \frac{u'}{\sqrt{N}}\zeta J_2(2\zeta) + \frac{(u')^2}{2N}\zeta^2 J_3(2\zeta) + \ldots$$
$$= F_0 + \frac{u'}{\sqrt{N}} F_1 + \frac{(u')^2}{N} F_2 + \ldots \quad (70)$$

$$e^{-N(u-\log u)}\frac{J_1(2\zeta\sqrt{u})}{\sqrt{u}}$$
$$= F_0 + \frac{1}{\sqrt{N}}\left(\frac{(u')^3}{3}F_0 + u'F_1\right)$$
$$+\frac{1}{N}\left(\left(\frac{(u')^6}{18} - \frac{(u')^4}{4}\right)F_0 + \frac{(u')^4}{3}F_1 + (u')^2 F_2\right). \quad (71)$$

Using this expansion we get

$$\rho(\zeta/N) = -2\zeta J_1(2\zeta)J_3(2\zeta) + J_1(2\zeta)J_2(2\zeta) + \zeta J_2^2(2\zeta)$$
$$= 2\zeta\left(J_1^2(2\zeta) - J_0(2\zeta)J_2(2\zeta)\right). \quad (72)$$

in agreement with (10) for $N_F = 1$.

7. Microscopic Limit : Massive Case

Let us consider the general case of one massive flavor. The inclusion of the mass corresponds to analyzing (52). Three different regimes may be be considered, depending on the strength of the sea quark mass. The regime $Nm >> 1$, corresponds to the case where the sea mass is comparable to the valence mass, in the physical (chiral) limit. In this case, the mass term in (52) contributes to the saddle point equations along s and $s^\dagger$. The saddle point equations in x and y remain unaffected, and the resulting spectral distribution in the large N limit is a semi-circle. This result is in agreement with the Coulomb gas analysis. We expect, the microscopic spectral density to be unaffected by the mass effect. In the regime $Nm << 1$, the mass effects are sub-leading compared to the $\mathcal{O}(1)$ terms. This regime corresponds to the massless case discussed above. Both the macroscopic and microscopic spectral densities are expected to be unaffected by such masses. The most interesting regime, is the one for which $Nm \sim 1$, as we now discuss.

In the case $\mu = Nm = \mathcal{O}(1)$,

$$G(z,m) = \mathcal{N}' \int_0^\infty dx\ dy\ ds_{ij}\ ds_{ij}^*(x+y)(xy)^{N-1} \quad (73)$$
$$\times\, e^{-Nxy - N\mathrm{tr} ss^\dagger} e^{i\zeta(x+y-s_{11}-s_{11}^*)} e^{-\mu(s_{22}+s_{22}^*)}$$
$$\times\, e^{(\zeta)^2/N} \det \mathbf{\Delta}(\mathbf{s}\,\mathbf{s}^\dagger, xy).$$

We can use the same parameterization as before and observe that

$$s_{22} + s_{22}^* = 2\sqrt{v}\sin\theta\cos(\beta+\psi) + 2\sqrt{w}\cos\theta\cos(\gamma+\psi). \quad (74)$$

Let us introduce

$$t = \cos\theta. \quad (75)$$

We have

$$G(z,m) = \mathcal{N}' \int_0^\infty \cosh\omega d\omega \int_0^\infty \frac{dp}{\sqrt{p}} \quad (76)$$
$$\times \int_0^\infty ududvdw \int_0^{2\pi} d\phi d\psi d\beta d\gamma$$
$$\times \int_0^1 t\ dtp^N (uw)^{N-1}(p^2 - p(u+w+v) + uw)$$
$$\times\, e^{-N(u+w+v+p)} e^{2i\zeta\sqrt{p}\cosh\omega} e^{2i\zeta\sqrt{u}t\cos\phi}$$
$$\times\, e^{-\mu\sqrt{w}t\cos\gamma - \mu\sqrt{v}\sqrt{1-t^2}\cos\beta}.$$

We perform the integrals over ω (as before) and over angular variables ϕ, ψ, β and γ using

$$\int_0^{2\pi} d\phi\ e^{ix\cos\phi} = 2\pi J_0(x), \quad (77)$$
$$\int_0^{2\pi} d\phi\ e^{-x\cos\phi} = 2\pi I_0(x). \quad (78)$$

We get

$$\rho(\zeta/N, \mu/N) = -\pi^2\sqrt{2}\left(\frac{N}{\pi}\right)^{7/2} e^{3N}\int_0^\infty dp\ dv\ du\frac{dw}{w}$$
$$\int_0^1 t\ dt(p^2 - p(u+w+v) + uw) \quad (79)$$
$$\times\, (puw)^N e^{-N(p+u+w+v)}\frac{J_1(2\zeta\sqrt{p})}{\sqrt{p}} J_0(2\zeta t\sqrt{u})$$
$$\times\, I_0(2\mu\sqrt{1-t^2}\sqrt{v}) I_0(2\mu t\sqrt{w}).$$

This is an explicit integral representation of the spectral distribution in the massive case and for finite N. No approximation has been used in getting from (74) to (80). We observe that the integral over v can be performed in the leading order and that to this order we can replace $I_0(2\mu\sqrt{1-t^2}\sqrt{v})$ by one. Using this fact and the formula:

$$\int_0^1 t\ dt J_0(xt) I_0(yt) = \frac{yI_1(y)J_0(x) + xJ_1(x)I_0(y)}{x^2+y^2} \quad (80)$$

we have

$$\rho(\zeta,\mu) \propto \int_0^\infty dp\ du \frac{dw}{w}(p^2 - p(u+w+\frac{1}{N}) + uw) \quad (81)$$
$$\times\, (puw)^N e^{-N(p+u+w)}\frac{J_1(2\zeta\sqrt{p})}{\sqrt{p}}$$
$$\times\, \frac{\mu\sqrt{w}I_1(2\mu\sqrt{w})J_0(2\zeta\sqrt{u}) + \zeta\sqrt{u}J_1(2\zeta\sqrt{u})I_0(2\mu\sqrt{w})}{2((\zeta)^2 u + \mu^2 w)}.$$

The integral is dominated again by the neighborhood of the saddle point at $p_c = u_c = w_c = 1$. We need as before terms $\mathcal{O}(1/N)$ to calculate it.

Expanding the above formula as before, we obtain after somewhat lengthy algebra,

$$\begin{aligned}
\rho(\zeta,\mu) = 2I_0 \Bigg(& J_0^2 \lambda \cos\tau - J_0 J_1 \frac{3+\cos 4\tau}{4} \\
& - J_1^2 \frac{-8\lambda^2\cos\tau + \cos 3\tau - \cos 5\tau}{8\lambda}\Bigg) \\
& - 2I_1 \sin\tau \Bigg(J_0^2 \cos\tau + J_0 J_1 \frac{\cos 2\tau \sin^2\tau}{\lambda} \\
& + 2J_1^2 \cos\tau \sin^2\tau \Bigg) . \qquad (82)
\end{aligned}$$

with the parameters

$$\begin{aligned}
\zeta &= \lambda\cos\tau, \qquad (83)\\
\mu &= \lambda\sin\tau.
\end{aligned}$$

The Bessel functions in (82) are defined as $J_n = J_n(2\zeta)$ and $I_n = I_n(2\mu)$. For $\tau = 0$, we recover (10) for $N_F = 1$ and (72).

Clearly, sea masses of the order $Nm = \mathcal{O}(1)$ do affect the character of the spectral oscillations near zero virtuality. This in turn would imply, new sum rules for the moment of the eigenvalues of the massive QCD Dirac operator. The very non-local character of the fermion determinant upsets the universality at zero virtuality.

The present analysis can be extended to arbitrary flavors $N_F = 2, 3, ...$ Indeed, for N_F flavors the bosonization of the fermion bilinears requires the introduction of $2(N_F+1)^2$ auxiliary fields, which we label generically by the matrices $\mathbf{s}$ and $\mathbf{s}^\dagger$. After integration, the analog of the determinant $\mathbf{\Delta}$ in (48) splits into $(2N_F+2)\times(2N_F+2)$ blocks each of size $N \times N$. By analogy with the one flavor case, the determinant can be reduced to the product of N determinants of size $(2N_F+2)\times(2N_F+2)$. The result, can be generically written in the form

$$\mathbf{\Delta}_{N_F} = P_{N_F+1}((a^2b^2), \mathbf{s}\mathbf{s}^\dagger)\mathrm{det}^{N-1}(\mathbf{s}\mathbf{s}^\dagger) \qquad (84)$$

where P is a polynomial of degree N_F+1 in a^2b^2 with coefficients depending on the invariants of the combination $\mathbf{s}\mathbf{s}^\dagger$. So the number of flavors enters explicitly as the degree of the polynomial and implicitly as the dimension of the matrix $\mathbf{s}$. In the large N limit the integral is dominated by the saddle point, in which the polynomial does not contribute. Hence, the saddle point configurations are the minima of the action (54), resulting into a semicircular distribution of eigenvalues. This result is totally in agreement with the Coulomb gas argument. We expect, the above arguments near zero virtuality to carry (tediously) to the massive case.

8. Conclusions

Using chiral random matrix models as inspired by QCD spin and flavor symmetries, we have discussed the effects of a light quark mass on the QCD Dirac spectrum. In the thermodynamical limit (large N), the quarks play a subleading role on the macroscopic spectral density, except near zero virtuality. For one flavor and three colors (GUE), we have shown that the microscopic spectral distribution is affected by quark masses $Nm = \mathcal{O}(1)$. The non-local character of the fermion determinant causes the spectral oscillations to be flavor and mass dependent near zero virtuality. New sum rules of the type discussed by Leutwyler and Smilga are therefore expected. Our arguments extend to more than one flavor.

Kalkreuter's spectral distributions from lattice SU(2) gauge theories make use of a hermitean variant of the Dirac operator. In the quenched approximation and for Wilson fermions, they can be understood using a Gaussian Orthogonal matrix model, reminiscent of the one used in quantum Hall fluids. In contrast to the preceding discussion, here the quark mass survives the thermodynamical limit. For one flavor, Kalkreuter's spectral distributions show that a spectral transition occurs for quarks masses of the order of the strange quark mass. The transition is reminiscent of the Mott-transition from metals to insulators, with the density of states at zero virtuality playing the role of an order parameter.

The similarity between the spectral distribution generated from lattice simulations and the one obtained from the chiral random matrix model with massive quarks suggests that the lattice results may be robust to cooling. If that is the case, then we suggest that a properly regulated quark condensate in the presence of finite current quark masses can be obtained from cooled lattice configurations, in agreement with the general lore of random matrix theory. This point is worth checking on the lattice.

The behavior of the spectral density near zero virtuality and for massive quarks, is of relevance to finite temperature lattice simulations of QCD [27]. The sensitivity of the lattice results both to the masses and finite volumes (whether temporal or spatial) in the infrared regime, are important for a reliable assessment of the chiral condensate, and hence the study of the spontaneous breaking of chiral symmetry in finite systems both in vacuum and matter. The role of the current quark masses and the number of flavors in the QCD phase transition is of paramount importance for issues such as the order and character of the transition. Some of these issues will be discussed elsewhere.

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